

SPECTRAL MAPPING THEOREMS AND PERTURBATION THEOREMS FOR BROWDER'S ESSENTIAL SPECTRUM

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Abstract. If T is a closed, densely defined linear operator in a Banach space, F. E. Browder has defined the essential spectrum of T , $\text{ess}(T)$ [1]. We derive below spectral mapping theorems and perturbation theorems for Browder's essential spectrum. If T is a bounded linear operator and f is a function analytic on a neighborhood of the spectrum of T , we prove that $f(\text{ess}(T)) = \text{ess}(f(T))$. If T is a closed, densely defined linear operator with nonempty resolvent set and f is a polynomial, the same theorem holds. For a closed, densely defined linear operator T and a bounded linear operator B which commutes with T , we prove that $\text{ess}(T+B) \subseteq \text{ess}(T) + \text{ess}(B) = \{\mu + \nu : \mu \in \text{ess}(T), \nu \in \text{ess}(B)\}$. By making additional assumptions, we obtain an analogous theorem for B unbounded.

Introduction. Let T be a closed, densely defined linear operator on a Banach space X . F. E. Browder [1] defined the essential spectrum of T , $\text{ess}(T)$, to be the set of $\lambda \in \sigma(T)$, the spectrum of T , such that at least one of the following conditions holds: (1) $R(\lambda - T)$, the range of $\lambda - T$, is not closed; (2) λ is a limit point of $\sigma(T)$; (3) $\bigcup_{r>0} N[(\lambda - T)^r]$ is infinite dimensional, where $N(A)$ denotes the null space of a linear operator A . Browder proved that $\lambda_0 \notin \text{ess}(T)$ if and only if $(\lambda - T)^{-1}$ is defined for $0 < |\lambda - \lambda_0| < \delta$ and the Laurent expansion of $(\lambda - T)^{-1}$ around λ_0 has only a finite number of nonzero coefficients with negative indices.

Recall that a closed, densely defined operator T is called a Fredholm operator if and only if $\dim N(T) < \infty$, $R(T)$ is closed, and $\text{codim } R(T) < \infty$. Let us write $G = \{\lambda \in C \mid \lambda - T \text{ is a Fredholm operator}\}$. Since G is known to be an open set let us denote by G_i , $i \in I$, the open connected components of G . Gohberg and Kreĭn [2] proved (though they never formally defined the essential spectrum) that $\lambda_0 \notin \text{ess}(T)$ if and only if λ_0 belongs to an open connected component G_i which has a nonempty intersection with the resolvent set of T .

There are several other characterizations of $\text{ess}(T)$. D. Lay [5] has shown that if C is a compact, bounded linear operator such that $C: D(T) \rightarrow D(T)$ and $(CT)(x) = (TC)(x)$ for $x \in D(T^2)$, then $\text{ess}(T+C) = \text{ess}(T)$ and that conversely $\text{ess}(T)$ is the largest subset of $\sigma(T)$ which remains invariant under such perturbations. Other authors have described the essential spectrum of T in terms of the ascent, descent, nullity and defect of T , but we shall not need these results.

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In this paper we shall find Lay's result especially useful. In the first section we shall establish spectral mapping theorems. If B is a bounded linear operator and f is a function analytic on a neighborhood of $\sigma(B)$, we shall prove that $\text{ess}(f(B)) = f(\text{ess}(B))$. If T is a closed, densely defined linear operator with nonempty resolvent and p is a polynomial, we shall show that $\text{ess}(p(T)) = p(\text{ess}(T))$.

In the second section we consider perturbation theorems for the essential spectrum. Let T be a closed, densely defined linear operator and B a bounded linear operator such that $B: D(T) \rightarrow D(T)$ and $(BT)(x) = (TB)x$ for $x \in D(T)$. Then we shall prove that $\text{ess}(T+B) \subset \text{ess}(T) + \text{ess}(B) = \{\nu + \mu \mid \nu \in \text{ess}(T), \mu \in \text{ess}(B)\}$. If $0 \notin \sigma(B)$ or if $0 \notin \text{ess}(B)$ and $\sigma(T) \neq C$ (either hypothesis implies that BT is closed), then we shall show that $\text{ess}(BT) \subset \text{ess}(B) \text{ess}(T) = \{\mu\nu \mid \mu \in \text{ess}(B), \nu \in \text{ess}(T)\}$. Finally, we shall establish a perturbation theorem which makes no boundedness assumption on B . Let T and B be closed, densely defined linear operators with $D(T) \subset D(B)$. Let $D = \{x \in D(T) \mid Tx \in D(B)\}$ and assume that $B: D \rightarrow D(T)$ and $(TB)x = (BT)x$ for $x \in D$. Assume that $T+B$ is closed, $\rho(T+B)$ is nonempty, and $\text{ess}(T)$ is compact. Then there is a compact subset S of $\text{ess}(B)$ such that $\text{ess}(T+B) \subset \text{ess}(T) + S = \{\nu + s \mid \nu \in \text{ess}(T), s \in S\}$.

I would like to thank Richard Beals for a number of helpful discussions about the last mentioned theorem. In particular he suggested a simplified proof of an early version of the theorem, and it is that proof which will be given here.

1. Spectral mapping theorems for the essential spectrum. We begin with some trivialities. It is clear from the definition of $\text{ess}(B)$ that if X is a finite dimensional vector space and $B: X \rightarrow X$ is a linear operator, then $\text{ess}(B)$ is empty. Thus we shall always assume that X is an infinite dimensional Banach space, since our theorems are trivial otherwise. In this case, if $B: X \rightarrow X$ is a bounded linear operator, $\text{ess}(B)$ is nonempty, since otherwise $\lambda - B$ would be Fredholm of index zero for all $\lambda \in C$, whence by Theorem 3.2 of [2], X would be finite dimensional.

THEOREM 1. *Let B be a bounded linear operator of a Banach space X into itself. Let f be an analytic function defined on an open neighborhood of $\sigma(B)$. Then $\text{ess}(f(B)) = f(\text{ess}(B))$.*

Proof. Let G be the open neighborhood of $\sigma(B)$ on which f is defined and let Γ be a Jordan curve (not necessarily connected) contained in G and enclosing $\sigma(B)$. We can select $\varepsilon_0 > 0$ such that Γ encloses $N_{\varepsilon_0}(\sigma(B)) = \{\lambda \mid d(\lambda, \sigma(B)) < \varepsilon_0\}$.

First, let us show that $\text{ess}(f(B)) \subset f(\text{ess}(B))$. To prove this it suffices to show that $\text{ess}(f(B)) \subset f(N_\varepsilon(\text{ess}(B)))$ for all ε , $\varepsilon_0 \geq \varepsilon > 0$. For suppose $\mu_0 \in f(N_\varepsilon(\text{ess}(B)))$ for all ε , $\varepsilon_0 \geq \varepsilon > 0$. Then we can find a sequence of points ν_n such that $d(\nu_n, \text{ess}(B)) \rightarrow 0$ and $f(\nu_n) = \mu_0$. Since $\text{ess}(B)$ is compact, we can find a convergent subsequence $\nu_{n_i} \rightarrow \nu_0 \in \text{ess}(B)$, and $f(\nu_0) = \mu_0$.

Thus, select ε , $\varepsilon_0 \geq \varepsilon > 0$. Since $\sigma(B) - N_\varepsilon(\text{ess}(B))$ is a compact set consisting of isolated points, let $\sigma(B) - N_\varepsilon(\text{ess}(B)) = \{\nu_1, \dots, \nu_m\}$. Let C_1, \dots, C_m be disjoint circles

about ν_1, \dots, ν_m and such that the only point of $\sigma(B)$ enclosed in C_i is ν_i . Write $P = (1/2\pi i) \sum_{j=1}^m \int_{C_j} (\nu - B)^{-1} d\nu$ and recall that P is a finite dimensional projection commuting with B (see, for instance, [2] or [4]). Since we can assume $\text{ess}(B)$ nonempty, let us select $\nu_0 \in \text{ess}(B)$ and set $B_1 = B - BP + \nu_0 P$. We claim that $\sigma(B_1) \subset N_\epsilon(\text{ess } B)$. To see this, set $M = (I - P)(X)$ and note that $B_1|_M = B|_M$ (B clearly maps M into itself). By the functional calculus for bounded linear operators, it is known that $\mu - B|_M$ is one-to-one and onto M for μ inside one of the C_i , and since the only other points in $\sigma(B)$ lie in $N_\epsilon(\text{ess } B)$, $\mu - B_1|_M$ is one-to-one and onto M for $\mu \notin N_\epsilon(\text{ess } B)$. On the other hand, if we write $V = PX$,

$$\mu - B_1|_V = (\mu - \nu_0)|_V,$$

so $\mu - B_1$ is one-to-one and onto V for $\mu \neq \nu_0$. It follows that $\mu - B_1$ is one-to-one and onto X for $\mu \notin N_\epsilon(\text{ess } B)$.

Our next claim is that $\text{ess}(f(B)) = \text{ess}(f(B_1))$. (Notice that $f(B_1)$ is defined since $\sigma(B_1) \subset N_\epsilon(\text{ess } B) \subset G$, and in fact Γ encloses $\sigma(B_1)$.) Our claim will follow from Lay's theorem if we can show that $K = f(B_1) - f(B)$ is compact and commutes with $f(B)$. Commutativity is clear. To see compactness notice that for $\mu \in \Gamma$ (so $\mu \in \rho(B) \cap \rho(B_1)$), $\mu - B$ and $\mu - B_1$ take M into M and V into V . Since $X = M \oplus V$, it follows that $(\mu - B)^{-1}$ and $(\mu - B_1)^{-1}$ take M into M and $(\mu - B)^{-1}x = (\mu - B_1)^{-1}x$ for $x \in M$. Thus, for $x \in X$, $(\mu - B_1)^{-1}x - (\mu - B)^{-1}x = (\mu - B_1)^{-1}(Px) - (\mu - B_1)^{-1}(Px)$, and we see finally that

$$f(B_1)x - f(B)x = \frac{1}{2\pi i} \int_{\Gamma} f(\mu)(\mu - B)^{-1}x d\mu = (f(B_1) - f(B))(Px),$$

a finite dimensional operator.

With the above observation we can complete the first half of the proof. We have $\text{ess}(f(B)) = \text{ess}(f(B_1)) \subset \sigma(f(B_1))$. But by the ordinary spectral mapping theorem, $\sigma(f(B_1)) = f(\sigma(B_1)) \subset f(N_\epsilon(\text{ess } B))$.

Conversely, we want to show that $\text{ess}(f(B)) \supset f(\text{ess } B)$. Assume not, so that there exists $\mu_0 \in f(\text{ess } B) - \text{ess } f(B)$. Since $\mu_0 \notin \text{ess } f(B)$, take a circle C about μ_0 which contains no other points of $\sigma(f(B))$. Let $P_1 = (1/2\pi i) \int_C (\mu - f(B))^{-1} d\mu$, a finite dimensional projection, and let $M_1 = (I - P_1)X$. It follows as above that for $\nu \in \Gamma$, $(\nu - B)^{-1}: M_1 \rightarrow M_1$ and thus $f(B)|_{M_1} = f(B|M_1)$. Also as above, we know that $\mu_0 - f(B)|_{M_1}$ is a one-to-one map of M_1 onto M_1 . Thus if $f(\nu_0) = \mu_0$ for $\nu_0 \in \text{ess}(B)$, $\nu_0 \notin \sigma(B|M_1)$, since if it were, we would have

$$\mu_0 \in \sigma(f(B|M_1)) = \sigma(f(B)|_{M_1}).$$

Now we can show that $\nu_0 \notin \text{ess}(B)$, which gives a contradiction. We use Browder's original definition. If we set $V_1 = P_1X$, the range of $\nu_0 - B$ is $M_1 \oplus (\nu_0 - B)V_1$, the span of a closed subspace and a finite dimensional subspace, hence closed. Since V_1 is finite dimensional $(\nu - B)|_{V_1}$ must be one-to-one and onto V_1 for

$0 < |\nu - \nu_0| < \delta$, and since $\nu_0 - B|M_1$ is one-to-one and onto M_1 , $\nu - B|M_1$ is one-to-one and onto M_1 for $|\nu - \nu_0| < r$, $r > 0$. Thus $\nu - B$ is one-to-one and onto for $0 < |\nu - \nu_0| < \min(r, \delta)$ and ν_0 is an isolated point of $\sigma(B)$. Finally, we clearly have $\dim(\bigcup_{j>0} N(\nu_0 - B)^j) \leq \dim V_1$. Q.E.D.

REMARK 1. Theorem 1 was proved by T. T. West [8] for the case that $\text{ess}(B) = \{0\}$ (the so-called "Riesz operators") and $f(0) = 0$.

If we try to generalize Theorem 1 to unbounded linear operators T , we have the problem of defining $f(T)$. If f is a polynomial, however, there is basically no problem and we can obtain an analogue of Theorem 1. First we need a lemma. This result is not difficult; a proof is given by Goldberg [3, pp. 107–108].

LEMMA 1 (GOLDBERG). *Let T be a closed, densely defined linear operator such that $\sigma(T) \neq C$. Let p be a polynomial of degree n ; then $p(T)$, with domain $D(T^n)$, is a closed, densely defined linear operator and $\sigma(p(T)) = p(\sigma(T))$.*

LEMMA 2. *Let T be a closed, densely defined linear operator on X such that $\sigma(T) \neq C$. Let F be a bounded, finite dimensional linear operator such that $(\nu - T)^{-1}F = F(\nu - T)^{-1}$ for $\nu \in \rho(T)$. Then (a) $F(X) \subset \bigcap_{j>0} D(T^j)$; (b) $(FT)x = (TF)x$ for $x \in D(T)$; (c) for all $j \geq 1$, $(F + T)^j x - T^j x = E_j x$ for $x \in D(T^j)$, E_j a bounded, finite dimensional linear operator such that $E_j: D(T^j) \rightarrow D(T^j)$ and $T^j E_j x = E_j T^j x$ for $x \in D(T^j)$; (d) for any polynomial p , $\text{ess}(p(T + F)) = \text{ess}(p(T))$.*

Proof. Select $\nu \in \rho(T)$. Since $D(T) = (\nu - T)^{-1}X$ and since $(\nu - T)^{-1}F = F(\nu - T)^{-1}$, $F: D(T) \rightarrow D(T)$. Showing $(FT)x = (TF)x$ for $x \in D(T)$ is equivalent to showing that $F(\nu - T)x = (\nu - T)Fx$ for $x \in D(T)$, and to prove this we simply apply $(\nu - T)^{-1}$ to both sides.

If we set $V = F(X)$, by assumption, V is finite dimensional. Since $D(T)$ is dense in X and F is bounded, $F(D(T))$ is a dense subspace of V and hence $F(D(T)) = V$, so that $V \subset D(T)$. Because $(FT)x = (TF)x$ for $x \in D(T)$, $T: V \rightarrow V$ and thus $\nu - T: V \rightarrow V$. However, V is finite dimensional, so we have $(\nu - T)^{-1}: V \rightarrow V$ and in fact $V = (\nu - T)^{-1}V$. It follows that $V = (\nu - T)^{-j}V$ for all $j \geq 1$, so $V \subset D(T^j)$ for all $j \geq 1$.

To prove (c), note that because $(TF)x = (FT)x$ for $x \in D(T)$, we can write $(T + F)^j x - T^j x = A_j Fx$ for $x \in D(T^j)$, A_j a linear operator defined on $D(T^j)$. However, by our above work, $V \subset D(T^j)$ and since V is finite dimensional, $\|A_j v\| \leq M \|v\|$ for $v \in V$. Thus we see that for $x \in X$, $\|A_j Fx\| \leq M \|Fx\| \leq M \|F\| \|x\|$, so that $E_j = A_j F$ is a bounded linear operator. By using the commutativity of F and T , we see that for $v \in D(T^j)$,

$$A_j v = \sum_{i=0}^{j-1} \binom{j}{i} T^i F^{j-i-1} v$$

and since $Fx \in D(T^{2j})$ for any x , it follows immediately that $A_j Fx \in D(T^j)$ for any x . Finally, since $T^k Fx = FT^k x$ for $x \in D(T^k)$ (by repeatedly using $(FT)x = (TF)x$ for $x \in D(T)$), it is easy to see that $T^j E_j x = E_j T^j x$ for $x \in D(T^j)$.

By using (c), we see that $p(T+F)x - p(T)x = Ex$ for $x \in D(T^n)$ ($n = \text{degree of } p$), E a finite dimensional, bounded linear operator such that $E: D(T^n) \rightarrow D(T^n)$ and $p(T)Ex = Ep(T)x$ for $x \in D(T^n)$. It follows by Lay's theorem that

$$\text{ess } (p(T+F)) = \text{ess } (p(T)).$$

THEOREM 2. *Let T be a closed, densely defined linear operator on a Banach space X and assume that $\sigma(T) \neq C$. Then if p is a polynomial, $p(T)$ is closed and $\text{ess } (p(T)) = p(\text{ess } (T))$.*

Proof. The proof follows the basic outlines of Theorem 1, so we shall be sketchy.

Let $V_R(0)$ denote the closed ball of radius R about 0 in C and let $V'_R(0)$ denote the complement of $V_R(0)$. We want to show that $\text{ess } (p(T)) \subset p(\text{ess } T)$ and in order to prove this it suffices to prove that

$$\text{ess } (p(T)) \subset p(N_\varepsilon(\text{ess } T) \cup V'_R(0)) \quad \text{for all } \varepsilon > 0, R > 0.$$

For suppose we have established the above inclusion and $\lambda_0 \in \text{ess } (p(T))$. Unless p is constant, in which case the theorem is trivial, there exists an R so large that $p(V'_R(0)) \subset V'_{|\lambda_0|}(0)$, so that $\lambda_0 \in p(N_\varepsilon(\text{ess } T) \cap V_R(0))$. If we select $\varepsilon_n \rightarrow 0$ and $\mu_n \in N_{\varepsilon_n}(\text{ess } T) \cap V_R(0)$ such that $p(\mu_n) = \lambda_0$, then since $\mu_n \in V_R(0)$, a compact set, we can assume $\mu_n \rightarrow \mu \in \text{ess } (T)$ and $\lambda_0 = p(\mu)$.

Consider

$$\begin{aligned} & (\sigma(T) - N_\varepsilon(\text{ess } T)) \cap V_R(0) \\ &= \{\mu_1, \dots, \mu_m\} \quad (\sigma(T) - N_\varepsilon(\text{ess } T)) = \{\lambda \mid \lambda \in \sigma(T), \lambda \notin N_\varepsilon(\text{ess } T)\}. \end{aligned}$$

If we let C_1, \dots, C_m be disjoint circles such that μ_j is the only point of $\sigma(T)$ contained in C_j , and if we set $P = (1/2\pi i) \sum_{j=1}^m \int_{C_j} (\lambda - T)^{-1} d\lambda$, then just as before P is a finite dimensional projection and $P: X \rightarrow D(T)$. If we select μ_0 with $|\mu_0| > R$ and if we set $F = -TP + \mu_0 P$, the same proof as before shows that $\sigma(T+F) \subset N_\varepsilon(\text{ess } T) \cup V'_R(0)$. Since P is finite dimensional, F is finite dimensional; and since P is continuous as a map from X to $D(T)$ with the graph topology and T is continuous from $D(T)$ with the graph topology to X , F is bounded. It is also clear that for $v \in \rho(T)$, $(v - T)^{-1}F = F(v - T)^{-1}$. It follows by Lemma 2 that

$$\text{ess } (p(T)) = \text{ess } (p(T+F)) \subset \sigma(p(T+F)) = p(\sigma(T+F)) \subset p(N_\varepsilon(\text{ess } T) \cup V'_R(0)).$$

The proof that $p(\text{ess } T) \subset \text{ess } (p(T))$ is essentially the same as in Theorem 1, and we omit it. We just note that a little more care must be exercised since T is not everywhere defined. Q.E.D.

2. Perturbation theorems for the essential spectrum. To begin this section let us establish some notation. We shall say that linear operators T and B satisfy hypothesis one (H_1) if T is a closed densely defined linear operator with domain $D(T)$, B is a bounded linear operator such that $B: D(T) \rightarrow D(T)$, and $(TB)(x) = (BT)(x)$ for $x \in D(T^2)$. Condition H_1 is Lay's definition of " T commutes with

B'' [5]. We shall say that linear operators T and B satisfy H_2 if T and B are the same as above, but we strengthen the commutativity condition to $(TB)(x) = (BT)(x)$ for $x \in D(T)$.

Our next lemma is concerned with the behavior of $\sigma(T+B)$ when T and B satisfy H_1 . In the case that T is bounded, this lemma is well known and follows by Banach algebra techniques. In the general case the lemma has been proved by J. T. Marti [6].

LEMMA 3 (MARTI). *Let T and B be linear operators satisfying H_1 . Then $T+B$ is closed and $\sigma(T+B) \subset \sigma(T) + \sigma(B) = \{\mu + \nu \mid \mu \in \sigma(B), \nu \in \sigma(T)\}$. If $\sigma(T)$ is empty, we interpret $\sigma(T) + \sigma(B)$ as the empty set.*

The following lemma is also proved by Marti [6]. If T is bounded, it follows by Banach algebra techniques without the assumption that $0 \notin \sigma(B)$.

LEMMA 4 (MARTI). *Let T and B be linear operators satisfying H_2 . Assume that $0 \notin \sigma(B)$. Then BT is a closed linear operator, and*

$$\sigma(BT) \subset \sigma(B)\sigma(T) = \{\mu\nu \mid \mu \in \sigma(B), \nu \in \sigma(T)\}.$$

If $\sigma(T)$ is empty, $\sigma(B)\sigma(T)$ is interpreted as empty.

REMARK 2. Marti actually proves Lemma 3 under the assumption that $B: D(T) \rightarrow D(T)$ and $(TB)x = (BT)x$ for $x \in D(T)$. However, if $\sigma(T) = C$, Lemma 3 is immediate. If $\sigma(T) \neq C$ and $\nu \in \rho(T)$, H_1 implies that $B(\nu - T)^{-1}x = (\nu - T)^{-1}Bx$ for $x \in D(T)$ (apply $(\nu - T)$ to both sides), whence by continuity $B(\nu - T)^{-1}x = (\nu - T)^{-1}Bx$ for all x . If $u \in D(T)$, $u = (\nu - T)^{-1}x$, so that

$$(\nu - T)Bu = (\nu - T)B(\nu - T)^{-1}x = Bx = B(\nu - T)u,$$

and Marti's commutativity condition holds.

REMARK 3. Marti considers closed, densely defined linear operators T and S such that $D(T) \subset D(S)$, $S: D(T) \rightarrow D(T)$, and $(TS)x = (ST)x$ for all $x \in D(T)$ such that $Tx \in D(S)$. This generality is illusory, however. For suppose $\rho(T) \neq \emptyset$ and select $\nu \in \rho(T)$. For $x \in D(S)$, $(\nu - T)S(\nu - T)^{-1}x = Sx$, by the commutativity condition. If we let Y denote the Banach space $D(T)$ with the graph topology $(\|x\|_Y = \|x\| + \|Tx\|)$, it is easy to see that S is a closed linear operator as a map from Y to Y . Thus S is bounded as a map from Y to Y . Since $(\nu - T)^{-1}: X \rightarrow Y$ is bounded and $(\nu - T): Y \rightarrow X$ is bounded, $\|(\nu - T)S(\nu - T)^{-1}x\| \leq M\|x\|$. It follows that $\|Sx\| \leq M\|x\|$ for $x \in D(S)$, and since S was assumed closed, $D(S) = X$ and S is bounded.

The same remark applies to Lay's article [5].

THEOREM 3. *Let T and B be linear operators satisfying H_1 . Then $T+B$ is closed and $\text{ess}(T+B) \subset \text{ess}(T) + \text{ess}(B) = \{\nu + \mu \mid \nu \in \text{ess}(T), \mu \in \text{ess}(B)\}$. If $\text{ess}(T)$ is empty, this means that $\text{ess}(T+B)$ is empty.*

Proof. If $\sigma(T)=C$, $\text{ess}(T)=C$ and we are done. Thus we assume $\sigma(T)\neq C$. Now we establish some simple commutativity relationships. By Remark 2 we know that $(TB)x=(BT)x$ for $x\in D(T)$. By the proof of Remark 2 we have $(\nu-T)^{-1}B=B(\nu-T)^{-1}$ for $\nu\in\rho(T)$, and it follows (by applying $\mu-B$ to both sides of the equation) that $(\mu-B)^{-1}(\nu-T)^{-1}=(\nu-T)^{-1}(\mu-B)^{-1}$ for $\mu\in\rho(B)$, $\nu\in\rho(T)$. The latter equation implies that $(\mu-B)^{-1}$ takes $D(T)$ into $D(T)$, so that we see (by applying $\mu-B$ to both sides of the equation) that $(\mu-B)^{-1}Tx=T(\mu-B)^{-1}x$ for $x\in D(T)$.

Select $\lambda_0\in\text{ess}(T+B)$. We want to show that $\lambda_0\in\text{ess}(T)+\text{ess}(B)$. Suppose we can show that for every $\varepsilon>0$, $\lambda_0\in N_\varepsilon(\text{ess } T)+N_\varepsilon(\text{ess } B)$. This will imply that for every $\varepsilon>0$, $\lambda_0\in N_{2\varepsilon}(\text{ess } T+\text{ess } B)$. Since $\text{ess } T$ is closed and $\text{ess } B$ is compact, $\text{ess } T+\text{ess } B$ is closed, and it will follow that $\lambda_0\in\text{ess } T+\text{ess } B$.

Thus take $\lambda_0\in\text{ess}(T+B)$ and select $\varepsilon>0$. Let R be a constant such that $|\lambda_0|<R$ and $|\mu|<R$ for $\mu\in N_\varepsilon(\sigma(B))$. As in §1, let $V_{2R}(0)=\{x\mid\|x\|\leq 2R\}$. Since spectral points which are not in the essential spectrum are isolated, we know that $\sigma(T)\cap(V_{2R}(0)-N_\varepsilon(\text{ess } T))=\{\nu_1,\dots,\nu_n\}$ and $\sigma(B)-N_\varepsilon(\text{ess } B)=\{\mu_1,\dots,\mu_m\}$. Let D_1,\dots,D_n be disjoint circles about ν_1,\dots,ν_n , each D_j containing no points of $\sigma(T)$ except ν_j . Similarly, let C_1,\dots,C_m be disjoint circles about μ_1,\dots,μ_m such that each C_j contains no point of $\sigma(B)$ except μ_j . As in §1 we know that

$$Q = \sum_{j=1}^m \frac{1}{2\pi i} \int_{C_j} (\mu-B)^{-1} d\mu$$

is a bounded, finite dimensional projection; similarly

$$P = \sum_{j=1}^n \frac{1}{2\pi i} \int_{D_j} (\nu-T)^{-1} d\nu$$

is a bounded, finite dimensional projection, and in fact $P: X\rightarrow D(T)$ is continuous from X to $D(T)$ with the graph topology.

We are now in a position to complete the proof. Take ν_0 , $|\nu_0|>2R$, and select $\mu_0\in\text{ess}(B)$ (since we always assume X infinite dimensional, $\text{ess}(B)\neq\emptyset$). Consider the bounded, finite dimensional linear operator $F=\nu_0P-TP+\mu_0Q-BQ$. By using the commutativity relations of the first paragraph of the proof, it is easy to see that for $\nu\in\rho(T)$, $(\nu-T)^{-1}F=F(\nu-T)^{-1}$. It follows by Lemma 2 that $F(X)\subset\bigcap_{j\geq 1}D(T^j)$ and $TFx=FTx$ for $x\in D(T)$. It is also clear that $FB=BF$, so that $(T+B)Fx=F(T+B)x$ for $x\in D(T+B)=D(T)$. It follows by Lay's theorem that $\text{ess}(T+B+F)=\text{ess}(T+B)$.

The next step is to make use of the form of F . We have

$$\text{ess}(B+T+F)\subset\sigma((B+\mu_0Q-BQ)+(T+\nu_0P-TP))=\sigma(B_1+T_1).$$

It is not hard to see that B_1 and T_1 satisfy H_1 . For instance, since

$$(\mu_0Q-BQ)(\nu-T)^{-1}=(\nu-T)^{-1}(\mu_0Q-BQ),$$

it follows by Lemma 2 that $\mu_0 Q - BQ: D(T) \rightarrow D(T)$ and $(\mu_0 Q - BQ)Tx = T(\mu_0 Q - BQ)x$ for $x \in D(T)$. Thus by Marti's theorem $\sigma(B_1 + T_1) \subset \sigma(B_1) + \sigma(T_1)$. However, by the same arguments used in the first section, $\sigma(T_1) \subset N_\varepsilon(\text{ess } T) \cup V'_{2R}(0)$ and $\sigma(B_1) \subset N_\varepsilon(\text{ess } B)$. Putting all this together, $\lambda_0 \in (N_\varepsilon(\text{ess } T) \cup V'_{2R}(0)) + N_\varepsilon(\text{ess } B)$. However, it is impossible that $\lambda_0 = \nu + \mu$ for $\nu \in V'_{2R}(0)$ and $\mu \in N_\varepsilon(\text{ess } B)$, since $\nu + \mu > 2R - R = R$ and $|\lambda_0| \leq R$. Thus $\lambda_0 \in N_\varepsilon(\text{ess } T) + N_\varepsilon(\text{ess } B)$. Since λ_0 was an arbitrary element of $\text{ess } (T+B)$, this shows that $\text{ess } (T+B) \subset N_\varepsilon(\text{ess } T) + N_\varepsilon(\text{ess } B)$. Q.E.D.

Notice that the above proof shows $\text{ess } (T+B)$ is empty if $\text{ess } (T)$ is empty.

Before we prove the corresponding theorem for $\text{ess } (BT)$, we need another lemma.

LEMMA 5. *Let B and T be linear operators satisfying H_2 . Assume that $\sigma(T) \neq C$ and $0 \notin \text{ess } (B)$. Then BT is a closed linear operator. Furthermore, there exists a bounded, finite dimensional linear operator F such that $B_1 = B - F$ is invertible, B_1 and T satisfy H_2 , $\text{ess } (B_1) = \text{ess } (B)$, and $\text{ess } (B_1 T) = \text{ess } (BT)$.*

Proof. If B is invertible, the lemma is immediate (even without the assumption that $\sigma(T) \neq C$). Thus assume $0 \in \sigma(B) - \text{ess } (B)$. Let C be a circle about 0 which contains no other points of $\sigma(B)$, and let $Q = (1/2\pi i) \int_C (\mu - B)^{-1} d\mu$, a bounded, finite dimensional projection. We define $F = BQ - Q$, and we have to show F satisfies the claims of the lemma.

The usual argument shows that $0 \notin \sigma(B - F)$, so that B_1 is invertible. It is easy to show that $(\nu - T)^{-1}F = F(\nu - T)^{-1}$ for $\nu \in \rho(T)$, so that by Lemma 2 we know that $F(X) \subset \bigcap_{j \geq 1} D(T^j)$ and $(TF)x = (FT)x$ for $x \in D(T)$. It follows in particular that since $F(X)$ is a finite dimensional subspace of $D(T)$, TF is actually a bounded, finite dimensional linear operator. Since B_1 is invertible, $B_1 T$ is closed and thus $BT = B_1 T + TF$, a sum of a closed and a bounded linear operator, is closed.

To see that B_1 and T satisfy H_2 , it suffices to show F and T satisfy H_2 , and we have already seen that $(FT)x = (TF)x$ for $x \in D(T)$. Trivially, $BF = FB$, so that by Lay's theorem, $\text{ess } (B_1) = \text{ess } (B)$. Finally, to show that $\text{ess } (B_1 T) = \text{ess } (BT)$, it suffices, by Lay's theorem, to show that $(BT)(TF)x = (TF)x(BT)x$ for $x \in D(T)$. However, $Fx \in \bigcap_{j \geq 1} D(T^j)$ for any x , so that $(TF)x \in D(T)$, and we can write $(BT)(TF)x = (TB)(TF)x$. If $x \in D(T)$, we also have $(TB)(TF)x = (TB)(FT)x = (TF)(BT)x$. Q.E.D.

THEOREM 4. *Let B and T satisfy H_2 . Assume that $0 \notin \text{ess } (B)$ and that BT is closed (under the hypotheses, Lemma 5 implies that BT is closed if $\sigma(T) \neq C$). Then $\text{ess } (BT) \subset \text{ess } (B) \text{ess } (T) = \{\nu\mu \mid \mu \in \text{ess } (B), \nu \in \text{ess } (T)\}$. If $\text{ess } (T)$ is empty, $\text{ess } (B) \text{ess } (T)$ is interpreted as being empty.*

Proof. Since $\text{ess } (B) \neq \emptyset$ and $0 \notin \text{ess } (B)$ (recall that we assume X infinite dimensional), we can select $\mu_0 \in \text{ess } (B)$, $\mu_0 \neq 0$. If $\sigma(T) = C$, then we have $\text{ess } (T) \text{ess } (B) \supset \{\lambda\mu_0 \mid \lambda \in C\} \supset C$, so that $\text{ess } (BT) \subset \text{ess } (B) \text{ess } (T)$. Thus we can

assume that $\sigma(T) \neq C$. By Lemma 5 we can find a bounded, invertible operator B_1 such that B_1 and T satisfy H_2 , $\text{ess}(B_1) = \text{ess}(B)$ and $\text{ess}(B_1 T) = \text{ess}(BT)$. Thus it suffices to show that $\text{ess}(B_1 T) \subset \text{ess}(B_1) \text{ess}(T)$. For notational convenience, we shall write B instead of B_1 , but now we can also assume B invertible.

Select $\lambda_0 \in \text{ess}(BT)$. Notice that to show $\lambda_0 \in \text{ess}(B) \text{ess}(T)$, it suffices to show that $\lambda_0 \in N_\varepsilon(\text{ess } B)N_\varepsilon(\text{ess } T)$ for all $\varepsilon > 0$. For suppose $\lambda_0 \in N_\varepsilon(\text{ess } B)N_\varepsilon(\text{ess } T)$ for all $\varepsilon > 0$. Since $\text{ess}(B)$ is closed and $0 \notin \text{ess}(B)$, we can assume that $|\mu| \geq \delta > 0$ for $\mu \in N_\varepsilon(\text{ess } B)$ and $0 < \varepsilon < \varepsilon_0$. For $0 < \varepsilon < \varepsilon_0$, it follows that if $\lambda_0 = \mu\nu$, $\mu \in N_\varepsilon(\text{ess } B)$ and $\nu \in N_\varepsilon(\text{ess } T)$, then $|\nu| \leq M = |\lambda_0|/\delta$. Thus (for n large enough), if we select $\mu_n \in N_{1/n}(\text{ess } B)$ and $\nu_n \in N_{1/n}(\text{ess } T)$ such that $\mu_n \nu_n = \lambda_0$, μ_n and ν_n lie in compact sets, and by taking subsequences we can assume $\mu_n \rightarrow \mu \in \text{ess}(B)$ and $\nu_n \rightarrow \nu \in \text{ess}(T)$. Thus we see that $\lambda = \mu\nu \in \text{ess}(B) \text{ess}(T)$.

We now proceed as in Theorem 3, though we shall omit many details. Take $\lambda_0 \in \text{ess}(BT)$. Select $\varepsilon > 0$ small enough so that $|\mu| \geq \delta > 0$ for $\mu \in N_\varepsilon(\text{ess } B)$. Select a constant $R > \lambda_0/\delta$ and a number ν_0 with $|\nu_0| \geq R$. Select $\mu_0 \in \text{ess}(B)$ as before. Just as in Theorem 3, we can find finite dimensional projections P and Q which correspond respectively to the eigenvalues $\{\nu_1, \dots, \nu_n\} = V_R(0) \cap (\sigma(T) - N_\varepsilon \text{ess}(T))$ and $\{\mu_1, \dots, \mu_m\} = \sigma(B) - N_\varepsilon(\text{ess } B)$. We want to use Lay's theorem to show that

$$\text{ess}(BT) = \text{ess}((B + \mu_0 Q - BQ)(T + \nu_0 P - TP)).$$

We write $F_1 = \mu_0 Q - BQ$ and $F_2 = \nu_0 P - TP$. It is easy to show that $(\nu - T)^{-1}F_1 = F_1(\nu - T)^{-1}$ and $(\nu - T)^{-1}F_2 = F_2(\nu - T)^{-1}$ for $\nu \in \rho(T)$. It follows by Lemma 2 that $F_i(X) \subset \bigcap_{j \geq 1} D(T^j)$, $(TF_i)x = (F_i T)x$ for $x \in D(T)$, and TF_i is a bounded, finite dimensional linear operator, $i = 1, 2$. Using these results, we see that if we write $(B + F_1)(T + F_2) = BT + F$, F is a bounded, finite dimensional linear operator and $(\nu - T)^{-1}F = F(\nu - T)^{-1}$ for $\nu \in \rho(T)$. Applying Lemma 2 again shows $F(X) \subset D(T)$ and $(TF)x = (FT)x$ for $x \in D(T)$. Since it is obvious that $FB = BF$, Lay's theorem now implies $\text{ess}(BT + F) = \text{ess}(BT)$.

Using the above results, we see that $B_2 = B + F_1$ and $T_2 = T + F_2$ satisfy the hypotheses of Lemma 4, so $\text{ess}(B_2 T_2) \subset \sigma(B_2 T_2) \subset \sigma(B_2)\sigma(T_2)$. Just as before, $\sigma(B_2) \subset N_\varepsilon(\text{ess } B)$ (so, in particular, B_2 is invertible), and $\sigma(T_2) \subset N_\varepsilon(\text{ess } T) \cup V'_R(0)$. By the selection of R , $\lambda_0 \neq \mu\nu$ for $\mu \in N_\varepsilon(\text{ess } B)$ and $\nu \in V'_R(0)$, so that $\lambda_0 \in N_\varepsilon(\text{ess } T)N_\varepsilon(\text{ess } T)$. As we have remarked, this establishes our theorem. Q.E.D.

Our next goal is to establish a perturbation theorem when B may not be bounded. To compensate for the unboundedness of B , we have to strengthen our assumptions on T .

THEOREM 5. *Let T and B be closed, densely defined linear operators with $D(T) \subset D(B)$. Let $D = \{x \in D(T) \mid Tx \in D(B)\}$ and assume $B: D \rightarrow D(T)$ and $(TB)x = (BT)x$ for $x \in D$. Assume that $T + B$ is closed and $\rho(T + B)$ is nonempty. Finally, suppose $\text{ess}(T)$ is compact. Then there exists a closed subspace X_1 of X and a bounded linear operator $B_1: X_1 \rightarrow X_1$ such that $\text{ess}(B_1) \subset \text{ess}(B)$ and $\text{ess}(T + B) \subset \text{ess}(T) + \text{ess}(B_1)$.*

Proof. For $v \in \rho(T)$ and $x \in D(B)$, we have $B(v-T)^{-1}x = (v-T)^{-1}Bx$, because $(v-T)^{-1}x \in D$ for $x \in D(B)$ and then $(v-T)B(v-T)^{-1}x = B(v-T)(v-T)^{-1}x = (v-T)(v-T)^{-1}Bx$. Since $\text{ess}(T)$ is compact, let C be a simple Jordan curve enclosing $\text{ess}(T)$ and lying in $\rho(T)$. Define $P = (1/2\pi i) \int_C (v-T)^{-1} dv$ and set $X_1 = PX$, $X_2 = (I-P)X$. As usual, $P: X \rightarrow D(T)$, and in fact since P is a projection, $P: X \rightarrow \bigcap_{j \geq 1} D(T^j)$. Since $(v-T)^{-1}Bx = B(v-T)^{-1}x$ for $x \in D(B)$, we find easily that $(BP)x = (PB)x$ for $x \in D(B)$. It follows that for $u \in X_1$, $Bu = BP^2u = PBPu \in X_1$. Thus we can define $B_1: X_1 \rightarrow X_1$ by $B_1u = Bu$, $u \in X_1$. Similarly, we define $T_1: X_1 \rightarrow X_1$. Since both B_1 and T_1 are closed and defined on all of X_1 , they are bounded. Similarly, we note that $B: X_2 \cap D(B) \rightarrow X_2$ and $T: X_2 \cap D(T) \rightarrow X_2$, so we define $B_2 = B|_{X_2 \cap D(B)}$ (viewed as a map into X_2) and $T_2 = T|_{X_2 \cap D(T)}$ (viewed as a map into X_2).

Standard results now imply that $\sigma(B) = \sigma(B_1) \cup \sigma(B_2)$, $\sigma(T) = \sigma(T_1) \cup \sigma(T_2)$ and $\sigma(T+B) = \sigma(T_1+B_1) \cup \sigma(T_2+B_2)$. Using these results and Browder's original definition of the essential spectrum, it is not hard to see that $\text{ess}(B) = \text{ess}(B_1) \cup \text{ess}(B_2)$, $\text{ess}(T) = \text{ess}(T_1) \cup \text{ess}(T_2)$ and $\text{ess}(T+B) = \text{ess}(T_1+B_1) \cup \text{ess}(T_2+B_2)$. We omit the proof. However, we know as usual that $\sigma(T_2)$ lies outside C , so that $\text{ess}(T_2)$ lies outside C . Since C encloses $\text{ess}(T)$, $\text{ess}(T_1) = \text{ess}(T)$ and $\text{ess}(T_2)$ is empty. Note also that $B_1T_1 = T_1B_1$. To show this, it suffices to show that $(BT)(Px) = (TB)(Px)$ for $x \in X$, and this is true since $Px \in D(T^2) \subset D$. It follows by Theorem 3 that $\text{ess}(T_1+B_1) \subset \text{ess}(T_1) + \text{ess}(B_1) = \text{ess}(T) + \text{ess}(B_1)$. Thus to complete the proof it suffices to show that $\text{ess}(T_2+B_2)$ is empty.

Since we assume $\rho(T+B)$ is nonempty, take $\mu_0 \in \rho(T+B)$, so that certainly $\mu_0 \in \rho(T_2+B_2)$. Similarly, take $\nu_0 \in \rho(T)$, so that $\nu_0 \in \rho(T_2)$. Then we can write

$$\begin{aligned} \text{ess}(T_2+B_2) &= \mu_0 + \text{ess}(-\mu_0 + T_2 + B_2) \\ &= \text{ess}((-\mu_0 + T_2 + B_2)(\nu_0 - T_2)^{-1}(\nu_0 - T_2)) + \mu_0. \end{aligned}$$

If we set $B_3 = (-\mu_0 + T_2 + B_2)(\nu_0 - T_2)^{-1}$, B_3 is a bounded, invertible operator. It is also easy to check that $B_3: D(T_2) \rightarrow D(T_2)$ and $B_3(\nu_0 - T_2)x = (\nu_0 - T_2)B_3x$ for $x \in D(T_2)$. It follows by Theorem 4 that $\text{ess}(B_3(\nu_0 - T_2)) \subset \text{ess}(B_3) \cup \text{ess}(\nu_0 - T_2)$. However, $\text{ess}(\nu_0 - T_2) = \nu_0 - \text{ess}(T_2)$ is empty, so that $\text{ess}(B_3(\nu_0 - T_2))$ is empty and $\text{ess}(T_2+B_2)$ is empty. Q.E.D.

COROLLARY 1. Let T and B be closed, densely defined linear operators with $D(T) \subset D(B)$. If $D = \{x \in D(T) \mid Tx \in D(B)\}$, assume that $B: D \rightarrow D(T)$ and $(TB)x = (BT)x$ for $x \in D$. Assume that $\text{ess}(T)$ is compact. Suppose that there exists a sequence $\lambda_n \in \rho(T)$, $|\lambda_n| \rightarrow \infty$, such that $\|(\lambda_n - T)^{-1}\| \leq C/|\lambda_n|$, C a constant. Finally, assume that for every $\varepsilon > 0$, there is a constant K_ε such that $\|Bx\| \leq \varepsilon\|Tx\| + K_\varepsilon\|x\|$ for $x \in D(T)$. Then the conclusion of Theorem 5 holds.

Proof. It is known (see [4]) that if $\|Bx\| \leq c\|Tx\| + K\|x\|$ for $x \in D(T)$ with $c < 1$, then $T+B$ is closed. Thus $T+B$ is certainly closed. To apply Theorem 5, it only remains to show that $\rho(T+B)$ is nonempty. We claim that $\lambda_n \in \rho(T+B)$ for n

large enough, and since $(\lambda_n - T - B) = (I - B(\lambda_n - T)^{-1})(\lambda_n - T)$, it suffices to show that $\|B(\lambda_n - T)^{-1}\| < 1$ for n large enough. But by our hypothesis,

$$\begin{aligned}\|B(\lambda_n - T)^{-1}x\| &\leq \varepsilon\|T(\lambda_n - T)^{-1}x\| + K_\varepsilon\|(\lambda_n - T)^{-1}x\| \\ &= \varepsilon\|\lambda_n(\lambda_n - T)^{-1}x - x\| + K_\varepsilon\|(\lambda_n - T)^{-1}x\| \\ &\leq \varepsilon\|x\| + \varepsilon C\|x\| + K_\varepsilon/|\lambda_n|\|x\|.\end{aligned}$$

Thus we merely select ε such that $\varepsilon + \varepsilon C < 1$ and then select n so large that

$$\varepsilon + \varepsilon C + K_\varepsilon/|\lambda_n| < 1. \quad \text{Q.E.D.}$$

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